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THE RIEMANN DELTA INTEGRAL ON TIME SCALES

Jae Myung Park*, Deok Ho Lee**, Ju Han Yoon***, Young Kuk Kim****, and Jong Tae Lim*****

Abstract. In this paper, we define the extension $f^* : [a, b] \to \mathbb{R}$ of a function $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$ for a time scale $\mathbb{T}$ and show that f is Riemann delta integrable on $[a, b]_{\mathbb{T}}$ if and only if f^* is Riemann integrable on $[a, b]$.

1. Introduction and preliminaries

Let $\mathbb{T}$ be a time scale, $a < b$ be points in $\mathbb{T}$, and $[a, b]_{\mathbb{T}}$ be the closed interval in $\mathbb{T}$. A partition $\mathcal{P} = \{(\xi_i, [t_{i-1}, t_i])\}_{i=1}^n$ of $[a, b]_{\mathbb{T}}$ is a collection of tagged intervals such that

$$a = t_0 < t_1 < \cdots < t_n = b, \quad t_i \in \mathbb{T} \text{ for each } i = 1, 2, \cdots, \text{n},$$

and ξ_i is an arbitrary point on $[t_{i-1}, t_i)_{\mathbb{T}}$.

Let f be a real-valued bounded function on $[a, b]_{\mathbb{T}}$. Let $M_i = \sup\{f(t) : t \in [t_{i-1}, t_i)_{\mathbb{T}}\}$ and $m_i = \inf\{f(t) : t \in [t_{i-1}, t_i)_{\mathbb{T}}\}$. The upper $\Delta-$sum $\overline{S}_{\mathcal{P}}(f)$ and the lower $\Delta-$sum $\underline{S}_{\mathcal{P}}(f)$ of f with respect to $\mathcal{P}$ are defined by

$$\overline{S}_{\mathcal{P}}(f) = \sum_{i=1}^n M_i(t_i - t_{i-1}), \quad \underline{S}_{\mathcal{P}}(f) = \sum_{i=1}^n m_i(t_i - t_{i-1}).$$

Let $\{(a_k, b_k)\}_{k=1}^\infty$ be the sequence of intervals contiguous to $[a, b]_{\mathbb{T}}$ in $[a, b]$.

For a function $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$, define the extension $f^* : [a, b] \to \mathbb{R}$ of f by

$$f^*(t) = \begin{cases} f(a_k) & \text{if } t \in (a_k, b_k) \text{ for some } k \\ f(t) & \text{if } t \in [a, b]_{\mathbb{T}}. \end{cases}$$

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Correspondence should be addressed to Young Kuk Kim, ykkim@dragon.seowon.ac.kr.

It is well-known [7] that $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is McShane delta integrable on $[a,b]_{\mathbb{T}}$ if and only if $f^* : [a,b] \to \mathbb{R}$ is McShane integrable on $[a,b]$.

In this paper, we consider the Riemann delta integral and show that a function $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is Riemann delta integrable on $[a,b]_{\mathbb{T}}$ if and only if $f^* : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$.

2. The Riemann delta integral

DEFINITION 2.1. For given $\delta > 0$, a partition $\mathcal{P} = \{(\xi_i, [t_{i-1}, t_i])\}_{i=1}^n$ is a δ-partition of $[a,b]_{\mathbb{T}}$ if for each $i \in \{1, 2, \cdots, n\}$ either $t_i - t_{i-1} \le \delta$ or $t_i - t_{i-1} > \delta$ and $\sigma(t_{i-1}) = t_i$, where $\sigma(t) = \inf\{s \in T : s > t\}$.

DEFINITION 2.2. A bounded function $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is Riemann delta integrable (or $R_\Delta-$integrable) on $[a,b]_{\mathbb{T}}$ if there exists a number A such that for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\Big|\sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}) - A\Big| < \epsilon$$

for every $\delta-$partition $\mathcal{P} = \{(\xi_i, [t_{i-1}, t_i])\}_{i=1}^n$ of $[a,b]_{\mathbb{T}}$. The number A is called the Riemann delta integral of f on $[a,b]_{\mathbb{T}}$ and we write

$$A = (R_\Delta)\int_a^b f.$$

The following theorem gives a Cauchy criterion for $R_\Delta-$integrability.

THEOREM 2.3. [3] *A bounded function $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is $R_\Delta-$integrable on $[a,b]_{\mathbb{T}}$ if and only if for each $\epsilon > 0$ there exists a partition $\mathcal{P}$ of $[a,b]_{\mathbb{T}}$ such that $\overline{S}_{\mathcal{P}}(f) - \underline{S}_{\mathcal{P}}(f) < \epsilon$.*

Let $\mathcal{P} = \{(\xi_i, [t_{i-1}, t_i])\}_{i=1}^n$ and $\mathcal{Q} = \{(\zeta_j, [x_{j-1}, x_j])\}_{j=1}^m$ be two partitions of $[a,b]$(or $[a,b]_{\mathbb{T}}$). If $\{t_0, t_1, \cdots, t_n\} \subset \{x_0, x_1, \cdots, x_m\}$, then we say that $\mathcal{Q}$ is a refinement of $\mathcal{P}$ and we denote $\mathcal{Q} \ge \mathcal{P}$.

Recall that $f : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$ with value A if for each $\epsilon > 0$ there exists a partition $\mathcal{P} = \{(\xi_i, [t_{i-1}, t_i])\}$ of $[a,b]$ such that

$$\Big|\sum_{j} f(\zeta_j)(x_j - x_{j-1}) - A\Big| < \epsilon$$

for every refinement $\mathcal{Q} = \{(\zeta_i, [x_{j-1}, x_j])\}$ of $\mathcal{P}$.

THEOREM 2.4. *A bounded function $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is $R_\Delta-$integrable on $[a,b]_{\mathbb{T}}$ if and only if $f^* : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$. In that case, $(R)\int_a^b f^* = (R_\Delta)\int_a^b f$.*

Proof. Let $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$ be $R_\Delta-$integrable on $[a, b]_{\mathbb{T}}$ and let $\epsilon > 0$. Then there exists a partition $\mathcal{P}_0 = \{(\xi_j^0, [t_{j-1}^0, t_j^0])\}_{j=1}^m$ of $[a, b]_{\mathbb{T}}$ such that

$$\Big| \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}) - (R_\Delta)\int_a^b f \Big| < \epsilon \tag{2.1}$$

for every partition $\mathcal{P} = \{(\xi_i, [t_{i-1}, t_i])\}_{i=1}^n \geq \mathcal{P}_0$ of $[a, b]_{\mathbb{T}}$.

Assume that $\mathcal{P}' = \{(\xi_i', [t_{i-1}', t_i'])\}_{i=1}^n$ is a partition of $[a, b]$ with $\mathcal{P}' \geq \mathcal{P}_0$, where we regard $\mathcal{P}_0$ as a partition of $[a, b]$.

If $i \leq n$, then there is a unique $j \leq m$ such that $[t_{i-1}', t_i'] \subseteq [t_{j-1}^0, t_j^0]$ and there is a $\xi_i'' \in [t_{j-1}^0, t_j^0]_{\mathbb{T}}$ with $f^*(\xi_i') = f(\xi_i'')$. For each $j \leq m$, there are $i_{1j}, i_{2j} \leq n$ such that $[t_{i_{1j}-1}', t_{i_{1j}}']$, $[t_{i_{2j}-1}', t_{i_{2j}}'] \subseteq [t_{j-1}^0, t_j^0]$ and

$$f(\xi_{i_{1j}}'') = \min_{[t_{i-1}', t_i'] \subseteq [t_{j-1}^0, t_j^0]} f(\xi_i''), \quad f(\xi_{i_{2j}}'') = \max_{[t_{i-1}', t_i'] \subseteq [t_{j-1}^0, t_j^0]} f(\xi_i'').$$

By (2.1), we have

$$\begin{aligned}
&\sum_{i=1}^{n} f^*(\xi_i')(t_i' - t_{i-1}') \\
&\qquad = \sum_{j=1}^{m} \sum_{[t_{i-1}', t_i'] \subseteq [t_{j-1}^0, t_j^0]} f(\xi_i'')(t_i' - t_{i-1}') \\
&\qquad = \sum_{j=1}^{m} \Big(\sum_{[t_{i-1}', t_i'] \subseteq [t_{j-1}^0, t_j^0]} f(\xi_i'') \frac{t_i' - t_{i-1}'}{t_j^0 - t_{j-1}^0} \Big)(t_j^0 - t_{j-1}^0) \\
&\qquad \leq \sum_{j=1}^{m} f(\xi_{i_{2j}}'')(t_j^0 - t_{j-1}^0) \\
&\qquad < \sum_{j=1}^{m} f(\xi_j^0)(t_j^0 - t_{j-1}^0) + 2\epsilon.
\end{aligned} \tag{2.2}$$

Similarly, we have

$$\sum_{i=1}^{n} f^*(\xi_i')(t_i' - t_{i-1}') > \sum_{j=1}^{m} f(\xi_j^0)(t_j^0 - t_{j-1}^0) - 2\epsilon. \tag{2.3}$$

From (2.1), (2.2), (2.3) we have

$$\begin{aligned}
&\Big|\sum_{i=1}^{n} f^*(\xi_i^{'})(t_i^{'} - t_{i-1}^{'}) - (R_\Delta)\int_a^b f\Big| \\
&\le \Big|\sum_{i=1}^{n} f^*(\xi_i^{'})(t_i^{'} - t_{i-1}^{'}) - \sum_{j=1}^{m} f(\xi_j^0)(t_j^0 - t_{j-1}^0)\Big| \\
&+ \Big|\sum_{j=1}^{m} f(\xi_j^0)(t_j^0 - t_{j-1}^0) - (R_\Delta)\int_a^b f\Big| \\
&< 2\epsilon + \epsilon = 3\epsilon.
\end{aligned}$$

Thus f^* is Riemann integrable on $[a,b]$ and $\int_a^b f^* = (R_\Delta)\int_a^b f$.

Conversely, suppose that $f^* : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$. Let $\epsilon > 0$. Then there exists a partition $\mathcal{P} = \{[x_i, y_i]\}_{i=1}^n$ of $[a,b]$ such that

$$\overline{S}_{\mathcal{P}}(f^*) - \underline{S}_{\mathcal{P}}(f^*) < \epsilon.$$

Let $\{(a_k, b_k)\}$ be the sequence of intervals contiguous to $[a,b]_{\mathbb{T}}$ in $[a,b]$. Put

$$\begin{aligned}
&A = \{i | [x_i, y_i] \subset [a_k, b_k] \text{ for some } k \in \mathbb{N}, i = 1, 2, \cdots, n\}, \\
&B = \{1, 2, \cdots, n\} - A.
\end{aligned}$$

We see that $[x_i, y_i]_{\mathbb{T}} \neq \emptyset$ for each $i \in B$. Put

$$s_i = \inf[x_i, y_i]_{\mathbb{T}}, \quad t_i = \sup[x_i, y_i]_{\mathbb{T}} \text{ for each } i \in B.$$

Put $B_1 = \{i \in B \mid x_i < s_i\}$, $B_2 = \{i \in B \mid t_i < y_i\}$
$B_3 = \{i \in B \mid s_i < t_i\}$.

Let $\mathrm{K} = \{k \in \mathbb{N} \mid [x_i, y_i] \subset [a_k, b_k] \text{ for some } i \in A\}$
$\cup \{k \in \mathbb{N} \mid [x_i, s_i] \subset [a_k, b_k] \text{ for some } i \in B_1\}$
$\cup \{k \in \mathbb{N} \mid [t_i, y_i] \subset [a_k, b_k] \text{ for some } i \in B_2\}$.

Then the partition

$$\begin{aligned}
\mathcal{P}^{'} = \{[x_i, y_i] \mid i \in A\} \cup \{[x_i, s_i] \mid i \in B_1\} \cup \{[t_i, y_i] \mid i \in B_2\} \\
\cup \{[s_i, t_i] \mid i \in B_3\}
\end{aligned}$$

is a refinement of $\mathcal{P}$. Hence, $\overline{S}_{\mathcal{P}'}(f^*) - \underline{S}_{\mathcal{P}'}(f^*) < \epsilon$.

Put $\mathcal{P}^{''} = \{[s_i, t_i] \mid i \in B_3\}$, $\mathcal{Q} = \{[a_k, b_k] \mid k \in K\} \cup \mathcal{P}^{''}$.

Then $\mathcal{Q}$ is a partition of $[a,b]_{\mathbb{T}}$ and

$$\begin{aligned}
\overline{S}_{\mathcal{Q}}(f) - \underline{S}_{\mathcal{Q}}(f) &= \overline{S}_{\mathcal{P}''}(f) - \underline{S}_{\mathcal{P}''}(f) \\
&= \overline{S}_{\mathcal{P}'}(f^*) - \underline{S}_{\mathcal{P}'}(f^*) < \epsilon.
\end{aligned}$$

By Theorem 2.3, f is $R_\Delta-$integrable on $[a,b]_{\mathbb{T}}$. □

THEOREM 2.5. *Let f be a bounded $R_\Delta-$integrable function on $[a,b]_{\mathbb{T}}$. Then f is $R_\Delta-$integrable on every subinterval $[c,d]_{\mathbb{T}}$ of $[a,b]_{\mathbb{T}}$.*

Proof. Let f be a bounded R_Δ-integrable function on $[a,b]_{\mathbb{T}}$. By Theorem 2.4, $f^* : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$. By the property of the Riemann integral, f^* is Riemann integrable on every subinterval $[c,d] \subset [a,b]$. By Theorem 2.4, f is $R_\Delta-$integrable on every subinterval $[c,d]_{\mathbb{T}} \subset [a,b]_{\mathbb{T}}$. □

THEOREM 2.6. *Let f and g be $R_\Delta-$integrable on $[a,b]_{\mathbb{T}}$ and α, β be real numbers. Then $\alpha f + \beta g$ is $R_\Delta-$integrable on $[a,b]_{\mathbb{T}}$ and*

$$(R_\Delta)\int_a^b (\alpha f + \beta g) = \alpha (R_\Delta)\int_a^b f + \beta (R_\Delta)\int_a^b g.$$

Proof. Let f and g be $R_\Delta-$integrable on $[a,b]_{\mathbb{T}}$. By Theorem 2.4, $\alpha f^* + \beta g^*$ is Riemann integrable on $[a,b]$ and

$$(R)\int_a^b (\alpha f^* + \beta g^*) = \alpha \ (R)\int_a^b f^* + \beta \ (R)\int_a^b g^*.$$

Hence, $\alpha f + \beta g$ is $R_\Delta-$integrable on $[a,b]_{\mathbb{T}}$ and

$$(R_\Delta)\int_a^b (\alpha f + \beta g) = \alpha \ (R_\Delta)\int_a^b f + \beta \ (R_\Delta)\int_a^b g.$$

□

THEOREM 2.7. *Let f be a bounded function on $[a,b]_{\mathbb{T}}$ and let $c \in \mathbb{T}$ with $a < c < b$. If f is $R_\Delta-$integrable on each of intervals $[a,c]_{\mathbb{T}}$ and $[c,b]_{\mathbb{T}}$, then f is $R_\Delta-$integrable on $[a,b]_{\mathbb{T}}$ and*

$$(R_\Delta)\int_a^b f = (R_\Delta)\int_a^c f + (R_\Delta)\int_c^b f.$$

Proof. If f is $R_\Delta-$integrable on $[a,c]_{\mathbb{T}}$ and $[c,b]_{\mathbb{T}}$, then f^* is Riemann integrable on $[a,c]$ and $[c,b]$. By the property of the Riemann integral, f^* is Riemann integrable on $[a,b]$ and

$$(R)\int_a^b f^* = (R)\int_a^c f^* + (R)\int_c^b f^*.$$

By Theorem 2.4, f is $R_\Delta-$integrable on $[a,b]_{\mathbb{T}}$ and

$$(R_\Delta)\int_a^b f = (R_\Delta)\int_a^c f + (R_\Delta)\int_c^b f.$$

□

THEOREM 2.8. *Let $\{f_n\}$ be a sequence of $R_\Delta-$integrable functions on $[a,b]_{\mathbb{T}}$ such that $f_n \to f$ uniformly on $[a,b]_{\mathbb{T}}$. Then f is $R_\Delta-$integrable on $[a,b]_{\mathbb{T}}$ and*

$$(R_\Delta)\int_a^b f = \lim_{n\to\infty}(R_\Delta)\int_a^b f_n.$$

Proof. Let $\{f_n\}$ be a sequence of $R_\Delta-$integrable functions on $[a,b]_{\mathbb{T}}$ such that $f_n \to f$ uniformly on $[a,b]_{\mathbb{T}}$. By Theorem 2.4, $\{f_n^*\}$ is a sequence of Riemann integrable functions on $[a,b]$ such that $f_n^* \to f^*$ uniformly on $[a,b]$.

By the property of Riemann integral, f^* is Riemann integrable on $[a,b]$ and

$$(R)\int_a^b f^* = \lim_{n\to\infty}(R)\int_a^b f_n^*.$$

By Theorem 2.4, f is $R_\Delta-$integrable on $[a,b]_{\mathbb{T}}$ and

$$(R_\Delta)\int_a^b f = \lim_{n\to\infty}(R_\Delta)\int_a^b f_n.$$

□

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Department of Mathematics
Chungnam National University
Daejeon 305-764, Republic of Korea
E-mail: parkjm@cnu.ac.kr

**
Department of Mathematics Education
KongJu National University
Kongju 314-701, Republic of Korea
E-mail: dhlee@kongju.ac.kr

***
Department of Mathematics Education
Chungbuk National University
Chungju 360-763, Republic of Korea
E-mail: yoonjh@cbnu.ac.kr

****
Department of Mathematics Education
Seowon University
Chungju 361-742, Republic of Korea
E-mail: ykkim@dragon.seowon.ac.kr

*****
Department of Mathematics
Chungnam National University
Daejeon 305-764, Republic of Korea
E-mail: shiniljt@gmail.com